CAMBRIDGE INTERNATIONAL EXAMINATIONS

GCE Advanced Level

MARK SCHEME for the October/November 2013 series

9709 MATHEMATICS

9709/31 Paper 3, maximum raw mark 75

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge will not enter into discussions about these mark schemes.

Cambridge is publishing the mark schemes for the October/November 2013 series for most IGCSE, GCE Advanced Level and Advanced Subsidiary Level components and some Ordinary Level components.

Mark Scheme Notes

Marks are of the following three types:

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

- When a part of a question has two or more "method" steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

- The symbol $\sqrt{}$ implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously "correct" answers or results obtained from incorrect working.

- Note: B2 or A2 means that the candidate can earn 2 or 0.
  B2/1/0 means that the candidate can earn anything from 0 to 2.

  The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.

- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking *g* equal to 9.8 or 9.81 instead of 10.

The following abbreviations may be used in a mark scheme or used on the scripts:

AEF Any Equivalent Form (of answer is equally acceptable)

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

BOD Benefit of Doubt (allowed when the validity of a solution may not be absolutely clear)

CAO Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)

CWO Correct Working Only – often written by a 'fortuitous' answer

ISW Ignore Subsequent Working

MR Misread

PA Premature Approximation (resulting in basically correct work that is insufficiently accurate)

SOS See Other Solution (the candidate makes a better attempt at the same question)

SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Penalties

MR –1 A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become "follow through √" marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR –2 penalty may be applied in particular cases if agreed at the coordination meeting.

PA –1 This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

1		Use correct quotient or product rule	M1	
		Obtain correct derivative in any form	A1	
		Justify the given statement	A1	**[3]**
2	*EITHER:*	State or imply non-modular equation $2^2\left(3^x-1\right)^2=\left(3^x\right)^2$, or pair of equations $2\left(3^x-1\right)=\pm 3^x$	M1	
		Obtain $3^x = 2$ and $3^x=\frac{2}{3}$ (or $3^{x+1} = 2$)	A1	
	OR:	Obtain $3^x = 2$ by solving an equation or by inspection	B1	
		Obtain $3^x=\frac{2}{3}$ (or $3^{x+1} = 2$) by solving an equation or by inspection	B1	
		Use correct method for solving an equation of the form $3^x = a$ (or $3^{x+1} = a$), where $a > 0$	M1	
		Obtain final answers 0.631 and –0.369	A1	**[4]**
3	*EITHER:*	Integrate by parts and reach $kx^{\frac{1}{2}}\ln x-m\int x^{\frac{1}{2}}.\frac{1}{x}\,\mathrm{d}x$	M1*	
		Obtain $2x^{\frac{1}{2}}\ln x-2\int\frac{1}{x^{\frac{1}{2}}}\,\mathrm{d}x$, or equivalent	A1	
		Integrate again and obtain $2x^{\frac{1}{2}}\ln x-4x^{\frac{1}{2}}$, or equivalent	A1	
		Substitute limits $x = 1$ and $x = 4$, having integrated twice	M1(dep*)	
		Obtain answer $4(\ln 4-1)$, or exact equivalent	A1	
	OR1:	Using $u = \ln x$, or equivalent, integrate by parts and reach $kue^{\frac{1}{2}u}-m\int e^{\frac{1}{2}u}\,\mathrm{d}u$	M1*	
		Obtain $2ue^{\frac{1}{2}u}-2\int e^{\frac{1}{2}u}\,\mathrm{d}u$, or equivalent	A1	
		Integrate again and obtain $2ue^{\frac{1}{2}u}-4e^{\frac{1}{2}u}$, or equivalent	A1	
		Substitute limits $u = 0$ and $u = \ln 4$, having integrated twice	M1(dep*)	
		Obtain answer $4\ln 4-4$, or exact equivalent	A1	
	OR2:	Using $u=\sqrt{x}$, or equivalent, integrate and obtain $ku\ln u-m\int u.\frac{1}{u}\,\mathrm{d}u$	M1*	
		Obtain $4u\ln u-4\int 1\,\mathrm{d}u$, or equivalent	A1	
		Integrate again and obtain $4u\ln u-4u$, or equivalent	A1	
		Substitute limits $u = 1$ and $u = 2$, having integrated twice or quoted $\int \ln u\,\mathrm{d}u$ as $u\ln u\pm u$	M1(dep*)	
		Obtain answer $8\ln 2-4$, or exact equivalent	A1	
	OR3:	Integrate by parts and reach $I=\frac{x\ln x\pm x}{\sqrt{x}}+k\int\frac{x\ln x\pm x}{x\sqrt{x}}\,\mathrm{d}x$	M1*	
		Obtain $I=\frac{x\ln x-x}{\sqrt{x}}+\frac{1}{2}I-\frac{1}{2}\int\frac{1}{\sqrt{x}}\,\mathrm{d}x$	A1	
		Integrate and obtain $I=2\sqrt{x}\ln x-4\sqrt{x}$, or equivalent	A1	
		Substitute limits $x = 1$ and $x = 4$, having integrated twice	M1(dep*)	
		Obtain answer $4\ln 4-4$, or exact equivalent	A1	**[5]**

4		Use correct product or quotient rule at least once	M1*	
		Obtain $\frac{dx}{dt} = e^{-t}\sin t - e^{-t}\cos t$ or $\frac{dy}{dt} = e^{-t}\cos t - e^{-t}\sin t$, or equivalent	A1	
		Use $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$	M1	
		Obtain $\frac{dy}{dx} = \frac{\sin t - \cos t}{\sin t + \cos t}$, or equivalent	A1	
		EITHER: Express $\frac{dy}{dx}$ in terms of tan t only	M1(dep*)	
		Show expression is identical to $\tan\left(t - \frac{1}{4}\pi\right)$	A1	
		OR: Express $\tan\left(t - \frac{1}{4}\pi\right)$ in terms of tan t	M1	
		Show expression is identical to $\frac{dy}{dx}$	A1	**[6]**
5	**(i)**	Use Pythagoras	M1	
		Use the sin2A formula	M1	
		Obtain the given result	A1	**[3]**
	(ii)	Integrate and obtain a $k\ln\sin\theta$ or $m\ln\cos\theta$ term, or obtain integral of the form $p\ln\tan\theta$	M1*	
		Obtain indefinite integral $\frac{1}{2}\ln\sin\theta - \frac{1}{2}\ln\cos\theta$, or equivalent, or $\frac{1}{2}\ln\tan\theta$	A1	
		Substitute limits correctly	M1(dep)*	
		Obtain the given answer correctly having shown appropriate working	A1	**[4]**
6	**(i)**	State or imply $AB = 2r\cos\theta$ or $AB^2 = 2r^2 - 2r^2\cos(\pi - 2\theta)$	B1	
		Use correct formula to express the area of sector ABC in terms of r and θ	M1	
		Use correct area formulae to express the area of a segment in terms of r and θ	M1	
		State a correct equation in r and θ in any form	A1	
		Obtain the given answer	A1	**[5]**
		[SR: If the complete equation is approached by adding two sectors to the shaded area above BO and OC give the first M1 as on the scheme, and the second M1 for using correct area formulae for a triangle AOB or AOC, and a sector AOB or AOC.]		
	(ii)	Use the iterative formula correctly at least once	M1	
		Obtain final answer 0.95	A1	
		Show sufficient iterations to 4 d.p. to justify 0.95 to 2 d.p., or show there is a sign change in the interval (0.945, 0.955)	A1	**[3]**

7 **(i)** State or imply partial fractions are of the form $\frac{A}{x-2}+\frac{Bx+C}{x^2+3}$ B1

Use a relevant method to determine a constant M1

Obtain one of the values $A=-1$, $B=3$, $C=-1$ A1

Obtain a second value A1

Obtain the third value A1 **[5]**

(ii) Use correct method to obtain the first two terms of the expansions of $(x-2)^{-1}$, $\left(1-\frac{1}{2}x\right)^{-1}$, $\left(x^2+3\right)^{-1}$ or $\left(1+\frac{1}{3}x^2\right)^{-1}$ M1

Substitute correct unsimplified expansions up to the term in x^2 into each partial fraction A1√+A1√

Multiply out fully by $Bx + C$, where $BC \neq 0$ M1

Obtain final answer $\frac{1}{6}+\frac{5}{4}x+\frac{17}{72}x^2$, or equivalent A1 **[5]**

[Symbolic binomial coefficients, e.g. $\begin{pmatrix}-1\\1\end{pmatrix}$ are not sufficient for the M1. The f.t. is on *A, B, C*.]

[In the case of an attempt to expand $\left(2x^2-7x-1\right)\left(x-2\right)^{-1}\left(x^2+3\right)^{-1}$, give M1A1A1 for the expansions, M1 for multiplying out fully, and A1 for the final answer.]

[If *B* or *C* omitted from the form of partial fractions, give B0M1A0A0A0 in **(i)**; M1A1√A1√ in **(ii)**]

8 **(a)** *EITHER:* Solve for u or for v M1

Obtain $u=\frac{2i-6}{1-2i}$ or $v=\frac{5}{1-2i}$, or equivalent A1

Either: Multiply a numerator and denominator by conjugate of denominator, or equivalent

Or: Set u or v equal to $x + iy$, obtain two equations by equating real and imaginary parts and solve for x or for y M1

OR: Using $a + ib$ and $c + id$ for u and v, equate real and imaginary parts and obtain four equations in a, b, c and d M1

Obtain $b + 2d = 2$, $a + 2c = 0$, $a + d = 0$ and $-b + c = 3$, or equivalent A1

Solve for one unknown M1

Obtain final answer $u = -2 - 2i$, or equivalent A1

Obtain final answer $v = 1 + 2i$, or equivalent A1 **[5]**

(b) Show a circle with centre $-i$ B1

Show a circle with radius 1 B1

Show correct half line from 2 at an angle of $\frac{3}{4}\pi$ to the real axis B1

Use a correct method for finding the least value of the modulus M1

Obtain final answer $\frac{3}{\sqrt{2}}-1$, or equivalent, e.g. 1.12 (allow 1.1) A1 **[5]**

9	**(i)**	*EITHER:*	Obtain a vector parallel to the plane, e.g. $\overrightarrow{AB} = -2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$	B1	
			Use scalar product to obtain an equation in *a, b, c,* e.g. $-2a + 4b - c = 0$, $3a - 3b + 3c = 0$, or $a + b + 2c = 0$	M1	
			Obtain two correct equations in *a, b, c*	A1	
			Solve to obtain ratio $a : b : c$	M1	
			Obtain $a : b : c = 3 : 1 : -2$, or equivalent	A1	
			Obtain equation $3x + y - 2z = 1$, or equivalent	A1	
		OR1:	Substitute for two points, e.g. *A* and *B*, and obtain $2a - b + 2c = d$ and $3b + c = d$	B1	
			Substitute for another point, e.g. *C*, to obtain a third equation and eliminate one unknown entirely from the three equations	M1	
			Obtain two correct equations in three unknowns, e.g. in *a, b, c*	A1	
			Solve to obtain their ratio, e.g. $a : b : c$	M1	
			Obtain $a : b : c = 3 : 1 : -2$, $a : c : d = 3 : -2 : 1$, $a : b : d = 3 : 1 : 1$ or $b : c : d = -1 : -2 : 1$	A1	
			Obtain equation $3x + y - 2z = 1$, or equivalent	A1	
		OR2:	Obtain a vector parallel to the plane, e.g. $\overrightarrow{BC} = 3\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$	B1	
			Obtain a second such vector and calculate their vector product e.g. $(-2\mathbf{i} + 4\mathbf{j} - \mathbf{k}) \times (3\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$	M1	
			Obtain two correct components of the product	A1	
			Obtain correct answer, e.g. $9\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$	A1	
			Substitute in $9x + 3y - 6z = d$ to find d	M1	
			Obtain equation $9x + 3y - 6z = 3$, or equivalent	A1	
		OR3:	Obtain a vector parallel to the plane, e.g. $\overrightarrow{AC} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$	B1	
			Obtain a second such vector and form correctly a 2-parameter equation for the plane	M1	
			Obtain a correct equation, e.g. $\mathbf{r} = 3\mathbf{i} + 4\mathbf{k} + \lambda(-2\mathbf{i} + 4\mathbf{j} - \mathbf{k}) + \mu(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$	A1	
			State three correct equations in x, y, z, λ, μ	A1	
			Eliminate λ and μ	M1	
			Obtain equation $3x + y - 2z = 1$, or equivalent	A1	**[6]**
	(ii)	Obtain answer $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, or equivalent		B1	**[1]**

(iii)	*EITHER:*	Use $\dfrac{\overrightarrow{OA}.\overrightarrow{OD}}{\left\|\overrightarrow{OD}\right\|}$ to find projection ON of OA onto OD	M1	
		Obtain $ON = \dfrac{4}{3}$	A1	
		Use Pythagoras in triangle OAN to find AN	M1	
		Obtain the given answer	A1	
	OR1:	Calculate the vector product of $\overrightarrow{OA}$ and $\overrightarrow{OD}$	M1	
		Obtain answer $6\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$	A1	
		Divide the modulus of the vector product by the modulus of $\overrightarrow{OD}$	M1	
		Obtain the given answer	A1	
	OR2:	Taking general point P of OD to have position vector $\lambda(\mathbf{i}+2\mathbf{j}+2\mathbf{k})$, form an equation in λ by either equating the scalar product of $\overrightarrow{AP}$ and $\overrightarrow{OP}$ to zero, or using Pythagoras in triangle OPA, or setting the derivative of $\left\|\overrightarrow{AP}\right\|$ to zero	M1	
		Solve and obtain $\lambda = \dfrac{4}{9}$	A1	
		Carry out method to calculate AP when $\lambda = \dfrac{4}{9}$	M1	
		Obtain the given answer	A1	
	OR3:	Use a relevant scalar product to find the cosine of AOD or ADO	M1	
		Obtain $\cos AOD = \dfrac{4}{9}$ or $\cos ADO = \dfrac{5}{3\sqrt{10}}$, or equivalent	A1	
		Use trig to find the length of the perpendicular	M1	
		Obtain the given answer	A1	
	OR4:	Use cosine formula in triangle AOD to find $\cos AOD$ or $\cos ADO$	M1	
		Obtain $\cos AOD = \dfrac{8}{18}$ or $\cos ADO = \dfrac{10}{6\sqrt{10}}$, or equivalent	A1	
		Use trig to find the length of the perpendicular	M1	
		Obtain the given answer	A1	[4]

10	(i)	State or imply $V = \pi h^3$	B1	
		State or imply $\dfrac{dV}{dt} = -k\sqrt{h}$	B1	
		Use $\dfrac{dV}{dt} = \dfrac{dV}{dh}.\dfrac{dh}{dt}$, or equivalent	M1	
		Obtain the given equation	A1	[4]
		[The M1 is only available if $\dfrac{dV}{dh}$ is in terms of h and has been obtained by a correct method.]		
		[Allow B1 for $\dfrac{dV}{dt} = k\sqrt{h}$ but withhold the final A1 until the polarity of the constant $\dfrac{k}{3\pi}$ has been justified.]		

(ii)	Separate variables and integrate at least one side	M1	
	Obtain terms $\frac{2}{5}h^{\frac{5}{2}}$ and $-At$, or equivalent	A1	
	Use $t = 0, h = H$ in a solution containing terms of the form $ah^{\frac{5}{2}}$ and $bt + c$	M1	
	Use $t = 60$, $h = 0$ in a solution containing terms of the form $ah^{\frac{5}{2}}$ and $bt + c$	M1	
	Obtain a correct solution in any form, e.g. $\frac{2}{5}h^{\frac{5}{2}} = \frac{1}{150}H^{\frac{5}{2}}t + \frac{2}{5}H^{\frac{5}{2}}$	A1	
(ii)	Obtain final answer $t = 60\left(1 - \left(\frac{h}{H}\right)^{\frac{5}{2}}\right)$, or equivalent	A1	**[6]**
(iii)	Substitute $h = \frac{1}{2}H$ and obtain answer $t = 49.4$	B1	**[1]**